doi:10.1006/jabr.2001.8820, available online at http://www.idealibrary.com on IDEAL®

Gröbner Bases for the Rings of Special Orthogonal and 2×2 Matrix Invariants[1]

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Communicated by Craig Huneke

Received October 27, 2000

We present a Gröbner basis for the ideal of relations among the standard generators of the algebra of invariants of the special orthogonal group acting on k-tuples of vectors. The cases of SO_3 and SO_4 are interpreted in terms of the algebras of invariants and semi-invariants of k-tuples of 2×2 matrices. In particular, we present in an explicit form a Gröbner basis for the 2×2 matrix invariants. Finally we use a Sagbi basis to show that the algebra of SO_2 invariants is a Koszul algebra. © 2001 Academic Press

Key Words: special orthogonal group; ring of invariants; Gröbner basis; standard tableaux.

1. INTRODUCTION

Throughout this paper, K denotes an infinite field of characteristic different from 2. We study the combinatorial structure of the algebra of

[1] Supported by OTKA No. F032325 and the Bolyai Research Fellowship.

0021-8693/01 $35.00

polynomial invariants of $SO(V)$, the special orthogonal group acting by the standard representation on k-tuples of vectors from an n-dimensional K-vector space V. Recall that the group $SO(V)$, sometimes denoted by $SO(n, K)$ or SO_n, consists of the linear transformations of V that preserve a nondegenerate symmetric bilinear form $\langle -, - \rangle : V \times V \to K$ and have determinant 1. Denote by $K[V^k]$ the algebra of polynomial functions on $V^k = V \times \cdots \times V$. The action $g \cdot (v_1, \dots, v_k) = (gv_1, \dots, gv_k)$ $(g \in SO(V), v_i \in V)$ induces an action on $K[V^k]$, and we put

$$R = K[V^k]^{SO(V)} = \{f \in K[V^k] \mid \forall g \in SO(V) : g \cdot f = f\}$$

for the algebra of polynomial invariants. Generators of R are described in [W2] when $\mathrm{char}(K) = 0$ and in [DP] for any odd characteristic (see also [R]). The algebra R has two types of generators. For $1 \le i \le j \le k$, consider the function $\langle i, j \rangle : V^k \to K$ defined by $\langle i, j \rangle : (v_1, \dots, v_k) \mapsto \langle v_i, v_j \rangle$. The second type occurs when $n \le k$. In this case, for $1 \le i_1 < \cdots < i_n \le k$, consider the function $|i_1 \cdots i_n| : V^k \to K$, where $|i_1 \cdots i_n| : (v_1, \dots, v_k) \mapsto |v_{i_1} \cdots v_{i_n}|$ and $|v_{i_1} \cdots v_{i_n}|$ denotes the determinant of the $n \times n$ matrix containing as columns the coordinate matrices of $v_{i_1}, \dots, v_{i_n}$ with respect to a fixed basis in V (so $|i_1 \cdots i_n|$ is determined up to a nonzero scalar, which depends on the choice of basis in V). The first fundamental theorem for $SO(V)$ invariants asserts that R is generated by the foregoing functions. We note that [DP] works with the standard scalar product on $V = K^n$. When the base field is algebraically closed, we can find an orthonormal basis in V, so we can assume that $\langle -, - \rangle$ is of this form. Otherwise, we pass to $\overline{K}[V^k]$, where $\overline{K}$ denotes the algebraic closure of K. Since K is infinite, the Zariski closure of $SO(n, K)$ in $GL(n, \overline{K})$ is $SO(n, \overline{K})$; hence $\overline{K}[V^k]^{SO(n,K)} = \overline{K}[V^k]^{SO(n,\overline{K})}$. For this latter algebra, we can apply the results of [DP] directly. It turns out that the generators are defined over K. Therefore, all conclusions also hold for $K[V^k]^{SO(V)}$.

Introduce the commutative polynomial algebra

$$P = K[x_{ij}, z_{i_1 \cdots i_n} | 1 \le i \le j \le k, 1 \le i_1 < \cdots < i_n \le k]$$

and the algebra homomorphism $\pi : P \to R$ defined by $\pi(x_{ij}) = \langle i, j \rangle$ and $\pi(z_{i_1 \cdots i_n}) = |i_1 \cdots i_n|$. The second fundamental theorem for $SO(V)$ invariants (cf. [W2] for $\mathrm{char}(K) = 0$ and [DP] for $\mathrm{char}(K) \neq 2$) gives the defining relations of the algebra of invariants or, equivalently, the generators of the kernel I of π as an ideal in P.

The knowledge of the defining relations of an algebra does not always allow one to immediately work with the elements of the algebra. For computational purposes, one needs the Gröbner basis of the corresponding ideal. In this paper we present a Gröbner basis of the ideal I of the relations among the $SO(V)$ invariants. Our work is closely related to and depends on the results of [C], who gave a Gröbner basis of the ideal generated by the $(n+1) \times (n+1)$ minors of a generic symmetric matrix. The corresponding factor ring can be identified with the ring of invariants of the full orthogonal group $O(V)$. Actually, a more general class of ideals is treated in [C], and it is interesting to note that these results turn out to help for our different purposes.

In Section 2 we state and prove our main result—Theorem 2.1, an explicit Gröbner basis of I. Our interest in the problem originates in the quantitative study of invariants of k-tuples of matrices under the simultaneous conjugation action of the general linear group or, more generally, in the study of polynomial invariants and semi-invariants of representations of quivers. In Section 3 we point out some implications of Theorem 2.1 on this topic. In particular, we present an explicit Gröbner basis for the 2×2 matrix invariants. In Section 4 we discuss the algebra of invariants of $SO(V)$ when $\dim(V) = 2$. We apply Sagbi bases, which are subalgebra analogues to Gröbner bases for ideals and which turn out to be useful in invariant theory. We show that the algebra of SO_2 invariants has a finite Sagbi basis with respect to a special choice of the nondegenerate quadratic form. As a consequence, we derive that the algebra of invariants is a Koszul algebra.

2. RELATIONS

We fix the pure lexicographic monomial order in P induced by the following order of the variables:

- $z_{i_1 \cdots i_n} < z_{j_1 \cdots j_n}$ if $i_s < j_s$ holds for the largest index $s \in \{1, \dots, n\}$ with $i_s \neq j_s$.
- $x_{11} < x_{12} < \cdots < x_{1k} < x_{22} < x_{23} < \cdots < x_{2k} < \cdots < x_{k-1,k-1} < x_{k-1,k} < x_{kk}$.
- $x_{ij} < z_{i_1 \cdots i_n}$ for any $1 \leq i \leq j \leq k$ and $1 \leq i_1 < \cdots < i_n \leq k$.

The *leading monomial* $\text{lead}(f)$ of $f \in P$ is the monomial which is maximal with respect to $<$ among the monomials with nonzero coefficient in f. We say that the subset $\{f_1, \dots, f_m\}$ of the ideal I of P is a *Gröbner basis* of I if for any nonzero $f \in I$ there exists an $f_s \in \{f_1, \dots, f_m\}$ such that $\text{lead}(f_s)$ divides $\text{lead}(f)$; i.e., $\text{lead}(f) = w \cdot \text{lead}(f_s)$ for an appropriate monomial w. In other words, the monomial ideal of P generated by the leading monomials of $f_1, \dots, f_m$ coincides with the vector subspace of P spanned by the leading monomials of all polynomials in I. In this case, $I = (f_1, \dots, f_m)$ holds; i.e., $f_1, \dots, f_m$ generate the ideal I and, as a vector space, the factor algebra P/I has a basis consisting of all monomials which are not divisible by some of the leading monomials of $f_1, \dots, f_m$ (see, e.g., [St2]).

To describe the defining relations of R, we need further terminology. Denote by X the $k \times k$ generic symmetric matrix whose (i, j) (and (j, i)) entry is x_{ij}, $1 \le i \le j \le k$. For $1 \le a_1 < \cdots < a_s \le k$ and $1 \le b_1 < \cdots < b_s \le k$, we put $\left|\begin{smallmatrix} a_1 \cdots a_s \\ b_1 \cdots b_s \end{smallmatrix}\right|$ for the determinant of the $s \times s$ minor of X obtained by choosing the entries in the crossings of the rows of index $a_1, \dots, a_s$ and the columns of index $b_1, \dots, b_s$. We ensure that the ordering of the x_{ij}'s satisfies the property that the leading monomial of $\left|\begin{smallmatrix} a_1 \cdots a_s \\ b_1 \cdots b_s \end{smallmatrix}\right|$ is equal to $x_{a_1 b_1} \cdots x_{a_s b_s}$, as required by Conca [C], and then we can apply his results. Note that for any $(v_1, \dots, v_k) \in V^k$, the rank of the Gram matrix $(\langle v_i, v_j\rangle)_{i,j=1}^k$ is at most n. Therefore, the identity $\det((\langle a_i, b_j\rangle)_{i,j=1}^{n+1}) = 0$ holds in R or, in other words, $\left|\begin{smallmatrix} a_1 \cdots a_{n+1} \\ b_1 \cdots b_{n+1} \end{smallmatrix}\right|$ is contained in I for any $1 \le a_1 < \cdots < a_{n+1} \le k$ and $1 \le b_1 < \cdots < b_{n+1} \le k$. Moreover, the Gram determinant satisfies the well-known identity $\det((\langle a_i, b_j\rangle)_{i,j=1}^{n}) = \beta \cdot |a_1 \cdots a_n| \cdot |b_1 \cdots b_n|$, where β is the determinant of the matrix of $\langle -, - \rangle$ with respect to the chosen basis in V. It follows that for any $1 \le a_1 < \cdots < a_n \le k$ and $1 \le b_1 < \cdots < b_n \le k$, the element

$$g_{b_1 \cdots b_n}^{a_1 \cdots a_n} = z_{a_1 \cdots a_n} \cdot z_{b_1 \cdots b_n} - \frac{1}{\beta} \det((x_{a_i b_j})_{i,j=1}^n) \in P$$

is contained in I.

Let H denote the set of strictly increasing sequences of integers from $\{1, \dots, k\}$. We introduce on H the following partial order: $a = (a_1, \dots, a_s) \preceq (b_1, \dots, b_t) = b$ if $\operatorname{length}(a) = s \ge \operatorname{length}(b) = t$ and $a_1 \le b_1, a_2 \le b_2, \dots, a_t \le b_t$. Now take $a, b, c \in H$ satisfying

$$\begin{aligned} \operatorname{length}(a) = n,\ \operatorname{length}(b) = {} & \operatorname{length}(c) \\ & = t \le n, a \preceq (b_1, \dots, b_{t-1}), a_t > b_t, b \preceq c. \end{aligned} \tag{1}$$

Observe that then $b_1 < b_2 < \cdots < b_t < a_t < a_{t+1} < \cdots < a_n$. Consider the symmetric group S_{n+1} acting on the set $\{b_1, \dots, b_t, a_t, \dots, a_n\}$ and its subset S_{n+1}^t consisting of the permutations which are monotone increasing on the subsets $\{b_1, \dots, b_t\}$ and $\{a_t, \dots, a_n\}$. (Obviously S_{n+1}^t has $\frac{(n+1)!}{t!(n+1-t)!}$ elements.) Define $h_{a,b,c} \in P$ by

$$h_{a,b,c} = \sum_{\sigma \in S_{n+1}^t} (-1)^\sigma z_{a_1 \cdots a_{t-1} \sigma(a_t) \cdots \sigma(a_n)} \begin{vmatrix} \sigma(b_1) \cdots \sigma(b_t) \\ c_1 \cdots c_t \end{vmatrix}.$$

Clearly, $\pi(h_{a,b,c})$ is alternating and multilinear in the arguments $v_{b_1}, \dots, v_{b_t}$, $v_{a_t}, \dots, v_{a_n}$, provided that the sets $\{b_1, \dots, b_t, a_t, \dots, a_n\}$ and $\{c_1, \dots, c_t, a_1, \dots, a_{t-1}\}$ are disjoint. Thus by elementary linear algebra, $\pi(h_{a,b,c}) = 0$ in this case, and specializing the arguments if necessary we get that $\pi(h_{a,b,c})$ is always 0. Now we are ready to formulate our main result.

THEOREM 2.1. *The following elements form a Gröbner basis of the ideal I of the defining relations of the algebra of $SO(V)$ invariants with respect to the monomial order introduced at the beginning of Section* 2:

- $\left|\begin{smallmatrix} a_1\cdots a_{n+1} \\ b_1\cdots b_{n+1}\end{smallmatrix}\right|$, $a \preceq b \in H$ *with* $\mathrm{length}(a) = \mathrm{length}(b) = n+1$
- $g^{a_1\cdots a_n}_{b_1\cdots b_n}$, $a, b \in H$ *with* $\mathrm{length}(a) = \mathrm{length}(b) = n$
- $h_{a,b,c}$, *where* $a, b, c \in H$ *satisfy* (1).

Remark. For comparison, we note that to get a system of generators of I, it is sufficient to take the elements $g^{a_1\cdots a_n}_{b_1\cdots b_n}$, $a, b \in H$ and $h_{a,b,c}$, $\mathrm{length}(b)=\mathrm{length}(c) = 1$, $a, b, c \in H$ satisfy (1) (see [Dr, Corollary 1.3]).

We make use of two kinds of natural gradings of the algebras R and P. The overgroup $O(V)$ of $SO(V)$ acts linearly on R, and $SO(V)$ is the kernel of this action. So we have a linear action of the two-element group $O(V)/SO(V)$ on R, and R decomposes as $R^0 + R^1$, where R^0 and R^1 are the eigenspaces with eigenvalues 1 and -1. This is a $\mathbb{Z}/(2)$-grading, which can be lifted to P by giving degree 1 to the z's and degree 0 to the x's.

The second grading is inherited from the usual $\mathbb{Z}$-grading of the polynomial algebra $K[V^k]$. Since the action of $SO(V)$ is homogeneous, R is a graded subalgebra of $K[V^k]$. To make π homogeneous, we define a new grading on the polynomial algebra P by putting $\deg(z_{i_1\cdots i_n}) = n$ and $\deg(x_{ij}) = 2$. Denote by R_d and P_d the degree d homogeneous component of R and P. The $\mathbb{Z}/(2)$-grading and the $\mathbb{Z}$-grading are compatible, so we have

$$P = \bigoplus_{d\geq 0}((P_d)^0 \oplus (P_d)^1) \text{ and } R = \bigoplus_{d\geq 0}((R_d)^0 \oplus (R_d)^1).$$

Recall the combinatorial description of R in the language of standard tableaux. To agree with the notation of [C] and [DP], we write the tableaux in the "francophone" way. By an *even tableau*, we mean a finite sequence $T = (b^1, c^1, b^2, c^2, \ldots, b^r, c^r)$, where $b^i, c^i \in H$, $\mathrm{length}(b^i) = \mathrm{length}(c^i)$ for $i = 1, \ldots, r$, and $\mathrm{length}(b^i) \geq \mathrm{length}(b^{i+1})$ for $i = 1, \ldots, r-1$. We say that T is *standard* if $b^1 \preceq c^1 \preceq b^2 \preceq c^2 \preceq \cdots \preceq b^r \preceq c^r$. The *degree* of T is $\deg(T) = \sum_{i=1}^r 2 \cdot \mathrm{length}(b^i)$, and the *length* of T is $\mathrm{length}(T) = \mathrm{length}(b^1)$.

By an *odd tableau*, we mean $T' = (a, T)$, where T is an even tableaux, and $a \in H$ with $\mathrm{length}(a) = n \geq \mathrm{length}(T)$. The degree of T' is $\deg(T') = n + \deg(T)$. The odd tableau T' is standard if T is standard and $a \preceq b^1$.

For each even and odd tableau (of length $\leq k$) we assign an element of P as follows: Let $T = (b^1, c^1, \ldots, b^r, c^r)$ be an even tableau; then set $u_T = \left|\begin{smallmatrix} b^1 \\ c^1\end{smallmatrix}\right| \cdots \left|\begin{smallmatrix} b^r \\ c^r\end{smallmatrix}\right|$. For an odd tableau $T' = (a, T)$, we set $u_{T'} = z_a \cdot u_T$.

It is proved in [DP] that

$$\{\pi(u_T) | T \text{ is a standard even tableaux}, \deg(T) = d,\ \mathrm{length}(T) \leq n\}$$

is a vector space basis of $(R_d)^0$ and

$$\{\pi(u_{T'}) | T' \text{ is a standard odd tableau, } \deg(T') = d\}$$

is a vector space basis of $(R_d)^1$. In particular,

$$\dim((R_d)^0) = \#\{\text{standard even tableaux of degree } d \text{ of length } \leq n\}$$

and

$$\dim((R_d)^1) = \#\{\text{standard odd tableaux of degree } d\}.$$

Proof of Theorem 2.1. We have already pointed out that the given elements are contained in I. Note that lead $(|{}^{a_1\cdots a_t}_{b_1\cdots b_t}|) = x_{a_1b_1}\cdots x_{a_tb_t}$, lead $(g^{a_1\cdots a_n}_{b_1\cdots b_n}) = z_{a_1\cdots a_n}z_{b_1\cdots b_n}$, and lead$(h_{a,b,c}) = z_{a_1\cdots a_n}x_{b_1c_1}\cdots x_{b_tc_t}$, since if for some permutation σ of the elements $b_1 < \cdots < b_t < a_{t+1} < \cdots < a_n$ there is a j with $\sigma(a_j) \in \{b_1, \ldots, b_t\}$, then $z_{a_1\cdots a_{t-1}\sigma(a_t)\cdots\sigma(a_n)} < z_{a_1\cdots a_n}$. Denote by J the ideal generated by $x_{a_1b_1}\cdots x_{a_{n+1}b_{n+1}}$ $(a \preceq b \in H)$, $z_{a_1\cdots a_n}z_{b_1\cdots b_n}$ $(a, b \in H)$, $z_{a_1\cdots a_n}x_{b_1c_1}\cdots x_{b_tc_t}$, where a, b, and c satisfy (1). Since the monomials in P are homogeneous with respect to both the $\mathbb{Z}$-grading and the $\mathbb{Z}/(2)$-grading, it is sufficient to show that

$$\#\{\text{monomials in } (P_d)^0\backslash J\} \leq \dim((R_d)^0)$$

and

$$\#\{\text{monomials in } (P_d)^1\backslash J\} \leq \dim((R_d)^1)$$

hold for any positive integer d. Observe that any monomial which has degree ≥ 2 in the z's is contained in J. Hence the monomials in $(P_d)^0\backslash J$ depend on the x_{ij}'s only. Hence the first of the foregoing inequalities is a restatement of [C, Theorem 2.9] which gives the Gröbner basis of the ideal of $K[X] = K[x_{ij} | 1 \leq i \leq j \leq k]$ generated by all minors $|{}^{a_1\cdots a_{n+1}}_{b_1\cdots b_{n+1}}|$. Now let us take a monomial from $(P_d)^1\backslash J$ of the form $z_{a_1\cdots a_n}w$, where w is a monomial in $K[X]$ not contained in the ideal of $K[X]$

$$L_a = (x_{b_1c_1}\cdots x_{b_tc_t} \mid t \leq n+1, b \preceq c, a \preceq (b_1, \ldots, b_{t-1}), a_t > b_t).$$

(Here we understand that $a_{n+1} = k+1$.) We can use the results of [C] to determine the number of degree $d-n$ monomials in L_a. Denote by J_a the ideal in $K[X]$ generated by $|{}^{b_1\cdots b_t}_{c_1\cdots c_t}|$ $(t \leq n+1, b \preceq c, a \preceq (b_1, \ldots, b_{t-1}), a_t > b_t)$. By [C, Theorem 2.8], the foregoing generating system of J_a is a Gröbner basis. It follows that L_a coincides with the ideal generated by the leading monomials of the elements of J_a, and the number of degree $d-n$ monomials of $K[X]$ not contained in L_a equals $\dim((K[X]/J_a)_{d-n})$. On the other hand, by [C, Lemma 2.6], we have that $\dim((K[X]/J_a)_{d-n})$ equals the number of standard even tableaux $T = (b^1, c^1, \dots, b^r, c^r)$ with $\deg(T) = d - n$ and $a \preceq b^1$. (Note that an even tableau satisfies these properties if and only if (a, T) is a standard odd tableau.)

Summarizing, we have shown that

$$\begin{aligned}\#\{\text{monomials in } (P_d)^1 \setminus J\} &\le \#\{\text{standard odd tableau of degree } d\} \\ &= \dim((R_d)^1),\end{aligned}$$

which completes the proof of our main result. ∎

3. MATRIX INVARIANTS

The results of Section 2 have relevance to the invariant theory of 2×2 matrices. Let $M_2^k = M_2 \times \cdots \times M_2$ denote the space of k-tuples of 2×2 matrices over K endowed with the simultaneous conjugation action of $GL_2(K)$:

$$g \cdot (A_1, \dots, A_k) = (gA_1g^{-1}, \dots, gA_kg^{-1})$$
$$(g \in GL_2, A_i \in M_2, i = 1, \dots, k).$$

The algebra $K[M_2^k]^{GL_2}$ is studied in several papers (see, e.g., [LB] and [Dr] and the references therein). Assume that K is algebraically closed. (Again, we see that the standard generators of $K[M_2^k]^{GL_2}$ are defined over the prime subfield of K, so our conclusions on the relations are valid for all K.) Then the given action of $GL_2(K)$ factors through $PGL_2(K) \cong SO(3, K)$ (see, e.g., [T, Theorem 11.6]), where SO_3 keeps the nondegenerate symmetric bilinear form $\langle A, B\rangle = \mathrm{tr}(AB)$, $A, B \in M_2$, invariant. We have an SO_3-module isomorphism $M_2 \cong K \oplus V$, where the SO_3-module K consists of all scalar matrices and $V = sl_2$ is the vector space of all traceless matrices. Hence K and V correspond, respectively, to the trivial representation and the standard three-dimensional representation of SO_3. It follows that $K[M_2^k]^{GL_2}$ is a k-variable polynomial algebra over $K[V^k]^{SO_3}$ generated by the linear functions $(A_1, \dots, A_k) \mapsto \mathrm{tr}(A_i)$, $i = 1, \dots, k$. Thus Theorem 2.1 gives a Gröbner basis for the ideal of relations of $K[M_2^k]^{GL_2}$. We write this basis explicitly.

Let $R' = K[M_2^k]^{GL_2}$ be the algebra of 2×2 matrix invariants. We present every 2×2 matrix A in the form $A = \frac{1}{2}\mathrm{tr}(A) + B$, where B is a traceless matrix. Hence the vector space M_2^k is a direct sum of $K^k = \{(\mathrm{tr}(A_1), \dots, \mathrm{tr}(A_k)) \mid A_i \in M_2\}$ and sl_2^k. We consider the polynomial algebra

$$P' = K[t_h, x_{ij}, u_{pqr} \mid 1 \le h \le k, 1 \le i \le j \le k, 1 \le p < q < r \le k].$$

It is well known (and follows from our discussion) that there is a surjective algebra homomorphism $\pi' : P' \to R'$ defined in the following way. The image $\pi'(t_h)$ of the variable t_h is the function which sends $(A_1, \ldots, A_k)$ to $\mathrm{tr}(A_h)$. The image of x_{ij} is the function $M_2^k \to K$ defined by $(A_1, \ldots, A_k) \mapsto \mathrm{tr}(B_iB_j)$, and the image of u_{pqr} is the function $M_2^k \to K$ defined by $(A_1, \ldots, A_k) \mapsto \mathrm{tr}([B_p, B_q]B_r)$, where $[B_p, B_q] = B_pB_q - B_qB_p$ (and $A_i = \frac{1}{2}\mathrm{tr}(A_i) + B_i$, $i = 1, \ldots, k$).

Fixing the orthogonal basis $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of sl_2, it may be obtained by direct calculation that the determinant of the triple $(B_p, B_q, B_r) \in sl_2^3$ satisfies

$$|B_pB_qB_r| = \frac{1}{4}\mathrm{tr}([B_p, B_q]B_r).$$

This allows to obtain as an immediate consequence of Theorem 2.1 the Gröbner basis of the ideal I' of P' equal to the kernel of π' with respect to the pure lexicographic monomial order in P' induced by the following order of variables:

- $t_1 < t_2 < \cdots < t_k$
- $u_{i_1i_2i_3} < u_{j_1j_2j_3}$ if $i_s < j_s$ holds for the largest index $s \in \{1, 2, 3\}$ with $i_s \neq j_s$
- $x_{11} < x_{12} < \cdots < x_{1k} < x_{22} < x_{23} < \cdots < x_{2k} < \cdots < x_{k-1,k-1} < x_{k-1,k} < x_{kk}$
- $t_h < x_{ij} < u_{i_1i_2i_3}$ for any $1 \le h \le k$, $1 \le i \le j \le k$ and $1 \le i_1 < i_2 < i_3 \le k$.

THEOREM 3.1. *The following elements form a Gröbner basis of the ideal I' of P' consisting of the defining relations of the algebra of GL_2 invariants of k-tuples of 2×2 matrices with respect to the monomial order introduced earlier*:

- $\left|\begin{smallmatrix} a_1a_2a_3a_4 \\ b_1b_2b_3b_4 \end{smallmatrix}\right| \stackrel{\text{def}}{=} \det(x_{a_ib_j})_{i,j=1}^4$, $a = (a_1, a_2, a_3, a_4) \preceq b = (b_1, b_2, b_3, b_4) \in H$
- $\left(g_{b_1b_2b_3}^{a_1a_2a_3}\right)' = u_{a_1a_2a_3}u_{b_1b_2b_3} + 2\det(x_{a_ib_j})_{i,j=1}^3$, $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in H$
- $h'_{a,b,c} \stackrel{\text{def}}{=} \sum_{\sigma\in S_4^t}(-1)^\sigma \; u_{a_1\cdots a_{t-1}\sigma(a_t)\cdots\sigma(a_3)} \; \det(x_{\sigma(b_i)c_j})_{i,j=1}^t$, where a, b, $c \in H$ satisfy (1) for $t \le n = 3$.

[Dr, Theorem 2.3] gives that a minimal set of defining relations of the algebra R' can be chosen among the relations $(g_{b_1b_2b_3}^{a_1a_2a_3})'$ and h'_{abc} for $t = 1$ which are, respectively, of degree 6 and 5 considered as polynomial functions on M_2^k. On the other hand, the Gröbner basis obtained in this paper consists of "additional" relations of degree 8 for $\left|\begin{smallmatrix} a_1a_2a_3a_4 \\ b_1b_2b_3b_4 \end{smallmatrix}\right|$ and of degree 7 and 9 for h'_{abc} and $t = 2, 3$.

Next we consider a closely related problem that fits into the more general framework of quiver representations. The group $SL(2, K) \times SL(2, K)$ also acts on M_2^k by

$$(g, h) \cdot (A_1, \dots, A_k) = (gA_1h^{-1}, \dots, gA_kh^{-1})$$
$$(g, h \in SL_2, A_i \in M_2, i = 1, \dots, k).$$

The algebra $K[M_2^k]^{SL_2\times SL_2}$ of semi-invariants of 2×2 matrices is used to construct a moduli space for representations of the quiver consisting of two vertices and k arrows from the first vertex to the second. We use this opportunity to point out that this algebra also allows an approach via the invariant theory of the special orthogonal group (which was not noticed by the first author when he wrote [Do]). Recall that $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ is the simply connected cover of $SO(4, \mathbb{C})$ (c.f. [OV, Chapter 4.3]). More generally, if K is algebraically closed, then the given representation of $SL_2 \times SL_2$ on M_2 factors through the standard four-dimensional representation of SO_4 (see, e.g., [T, Exercise 12.21]). (In particular, the action of $SL_2 \times SL_2$ preserves a nondegenerate quadratic form on M_2, the determinant.) Therefore, we have $K[M_2^k]^{SL_2\times SL_2} \cong K[V^k]^{SO_4}$, and [Do, Theorem 3.1] follows from [S] or [W1], where a rational expression of the Poincaré series of $\mathbb{C}[V^k]^{SO(V)}$ is computed for arbitrary n. This also shows that [Do, Corollary 5.1] holds for any infinite field K whose characteristic is different from 2. Furthermore, the case $n = 4$ of Theorem 2.1 can be interpreted in terms of semi-invariants of 2×2 matrices, and yields a Gröbner basis of the ideal of relations among the generators of $K[M_2^k]^{SL_2\times SL_2}$.

4. SAGBI BASES

The following two problems were asked by the referee, whom we thank for the inspiring report.

PROBLEM 4.1. Does $K[V^k]^{SO(V)}$ have a finite Sagbi basis?

We refer to [ST] for a survey on Sagbi bases in invariant theory. Note that by the remark after Theorem 2.1, the kernel of $\pi : P \to K[V^k]^{SO(V)}$ is generated as an ideal by quadratic polynomials if and only if $\dim(V) = 2$, so the following question makes sense (see, e.g., [HHR] for the notion of Koszul algebras).

PROBLEM 4.2. When $\dim(V) = 2$, is $K[V^k]^{SO_2}$ Koszul?

The motivation for posing Problem 4.2 here is that for an affirmative answer, it would be sufficient to show that $K[V^k]^{SO_2}$ can be defined by a Gröbner basis of polynomials of degree 2. Unfortunately, the Gröbner basis given by Theorem 2.1 contains elements of degree 3. On the other hand, the knowledge of a Sagbi basis of a subalgebra of a polynomial algebra can also help prove that the given algebra is Koszul (see[CHV]). It turns out that Problem 4.1 has an easy answer in the special case $\dim(V) = 2$, and this provides an affirmative answer to Problem 4.2.

For the rest of the paper, we assume that $\dim(V) = 2$, so $K[V^k] = K[x_i, y_i \mid i = 1, \dots, k]$, where x_i and y_i are the coordinate functions on the ith component of V^k. We identify SO_2 with the subgroup of $SL(V)$ preserving the nondegenerate quadratic form xy on V.

PROPOSITION 4.3. *In the foregoing notation, the algebra $K[V^k]^{SO_2}$ has a finite Sagbi basis and is Koszul.*

Proof. By the first fundamental theorem, $R = K[V^k]^{SO_2}$ is generated by $x_iy_i, x_iy_j + y_ix_j, x_iy_j - y_ix_j, (1 \le i, j \le k)$. Hence $R = K[x_iy_j \mid 1 \le i, j \le k]$ itself is an algebra generated by monomials. In particular, $\{x_iy_j \mid 1 \le i, j \le k\}$ is a finite Sagbi basis of R. Consider the homomorphism $\psi : K[t_{ij} \mid 1 \le i, j \le k] \to R$ mapping the generator t_{ij} of the k^2-variable polynomial algebra to x_iy_j. The determinants of the 2×2 minors of the $k \times k$ generic matrix (t_{ij}) form a Gröbner basis of the kernel of ψ (see, e.g., [St1]). As we noted earlier, this implies that R is Koszul.

ACKNOWLEDGMENTS

This research was done in the framework of the exchange program between the Hungarian and Bulgarian Academies of Sciences. We thank them for their support.

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